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Exact Solutions of the Boltzmann Equation in the VHP Model with Removal Interaction

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The linear and nonlinear Boltzmann equation for very hard particles (VHP) is considered in the case when the collision between two particles may lead not only to elastic scattering, but also to a removal event with the disappearance of the molecules. The extended transport equation is solved for arbitrary initial distributions. The computations are carried out explicitly for a special class of initial distributions and for various removal rates. The results are demonstrated graphically. Finally, source terms fulfilling physically reasonable conditions are introduced into the VHP model, and the time-dependent particle number is calculated.

KEY WORDS: Linear and nonlinear Boltzmann equation; VHP model; removal; external sources; exact solutions.

1. INTRODUCTION

Exact solutions of the nonlinear Boltzmann equation have been found only for special model cases.[1,2] Because of the complex structure of all physically relevant scattering kernels, this integrodifferential equation resists a strong solution in general. Particularly remarkable, therefore, is the discovery of an exact solution of the nonlinear Boltzmann equation for a spatially homogeneous and isotropic gas of Maxwell molecules by Krupp[3] and independently by Bobylev[4] and Krook and Wu.[5,6] This special solution, the BKW mode, holds only for a distinct class of initial distributions.

For arbitrary initial conditions, Ernst and Hendriks[7,8] obtained, by applying the Laplace transformation, a closed solution of the model-Boltzmann equation for a system of very hard particles (VHP) with two

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translational degrees of freedom. Though the differential cross section of the VHP model does not correspond to any physical interaction law, it satisfies the conservation laws (that of momentum only when interpreted as deterministic[9]) and a H-theorem can be derived for it. Thus the VHP-Boltzmann equation is not only of high interest as a closed soluble mathematical model. As a physical model, it is expected to predict too fast a relaxation in the high-energy tail of the distribution since the cross section, which increases (in contrast to the hard-sphere model[10]) like $(\text{energy})^{1/2}$, overestimates the efficiency of the collisions at high energies. The complete solubility of the VHP-Boltzmann equation enables the relaxation process to be analyzed for all energies and any nonuniformities in the approach to equilibrium to be studied. It is further important to note that there exists a simple nonlinear mapping between the VHP model and the kinetic equation of reacting polymeres.[11,12]

An essential goal of nonlinear rarefied gas dynamics is the molecular kinetic approach to chemical reactions (see ref. 13 and references therein). This requires not only the consideration of elastic and inelastic collisions, but also that removal events and external sources be taken into account. Hence, the intention of this paper is to extend the treatment of ref. 8 by including removal and external source terms in the VHP equation.

First, we consider in our model scattering and removal effects of test particles between themselves as well as removal effects of test particles when colliding with the field particles of a host medium. By resorting to the special case of constant collision frequencies for removal events, we obtained exact solutions for both the number densities and distribution functions due to arbitrary initial conditions. The essential step in the solution procedure rests upon the introduction of a new dependent variable in connection with an appropriate transformation of the time variable. With these new variables the extended VHP-Boltzmann equation appears in its simple original form and can be solved in a closed form by applying the method developed by Hendriks and Ernst.[8]

If in addition physically reasonable external source terms are taken into account, then it is only possible to calculate analytically the time-dependent particle number.

We further show that our method can be successfully applied to attack the linearized VHP problem for the case in which scattering and removal events occur.

Finally, we consider an initial distribution as a sum of two Maxwellians and compute explicitly the time evolution of the distribution function for various removal rates in the linear and nonlinear cases.

2. THE NONLINEAR VHP MODEL WITH REMOVAL EFFECTS

2.1. Formulation of the Kinetic Equation

The time evolution of the distribution function of a spatially homogenous and isotropic gas of colliding neutral particles is generally described by the nonlinear Boltzmann equation[1]

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = \iiint d\mathbf{w}\, d\mathbf{v}'\, d\mathbf{w}' [W(\mathbf{v}'\mathbf{w}' \mid \mathbf{v}\mathbf{w})\, f(\mathbf{v}', t)\, f(\mathbf{w}', t) - W(\mathbf{v}\mathbf{w} \mid \mathbf{v}'\mathbf{w}')\, f(\mathbf{v}, t)\, f(\mathbf{w}, t)] \tag{1}$$

In the very hard particle (VHP) model[8] in two dimensions, binary collisions $(\mathbf{v}, \mathbf{w} \to \mathbf{v}', \mathbf{w}')$ occur with the transition probability

$$W(\mathbf{v}\mathbf{w} \mid \mathbf{v}'\mathbf{w}') = W(\mathbf{v}'\mathbf{w}' \mid \mathbf{v}, \mathbf{w}) = \delta(v^2 + w^2 - v'^2 - w'^2) \tag{2}$$

It is further convenient to change to the energy representation of the distribution function

$$F(x, t) = 2\pi f(|\mathbf{v}|, t), \qquad x = \tfrac{1}{2} v^2 \tag{3}$$

Following the analysis of Hendriks and Ernst,[8] the VHP-Boltzmann equation is obtained (apart from a numerical factor which will be absorbed in the unit of time):

$$\frac{\partial}{\partial t} F(x, t) = \int_x^\infty du \int_0^u dy [F(y, t)\, F(u-y, t) - F(x, t)\, F(u-x, t)] \tag{4}$$

If only scattering events are considered, the particle number $N = \int dx\, F(x, t)$ and the total energy $E = \int dx\, xF(x, t)$ of the system are conserved.

Choosing units $N = E = 1$, it is then possible to rewrite the VHP equation (4) as

$$\left(\frac{\partial}{\partial t} + x + 1\right) F(x, t) = \int_x^\infty du \int_0^u dy\, F(y, t)\, F(u-y, t) \tag{5}$$

The solution of this equation has been obtained in closed form for arbitrary initial conditions by Ernst and Hendriks.[7]

Now we extend the VHP equation by introducing the removal term

$$\left(\frac{\partial f}{\partial t}\right)_{R_1} = \int d\mathbf{w}\, W(\mathbf{v}, \mathbf{w})\, f(\mathbf{v}, t)\, f(\mathbf{w}, t) \tag{6}$$

which describes the interaction of test particles assuming that in these events the colliding particles are removed.[14,15]

We confine ourselves to the simple but important case of constant removal collision frequencies $W(\mathbf{v}, \mathbf{w}) = C_R = \text{const}$:

$$\left(\frac{\partial f}{\partial t}\right)_{R_1} = C_R \cdot f(\mathbf{v}, t) \int d\mathbf{w} f(\mathbf{w}, t) = C_R f(\mathbf{v}, t)\, N(t) \tag{7}$$

where $N(t)$ denotes the time-dependent particle number.

The introduction of a further lost term

$$\left(\frac{\partial f}{\partial t}\right)_{R_2} = \hat{C}_R \hat{N} f(\mathbf{v}, t) \tag{8}$$

should describe the removal of test particles when colliding with field particles of a host medium of fixed total density $\hat{N}$. Taking into account Eqs. (7) and (8), then we can write the initial VHP-Boltzmann equation (4) in the form

$$\begin{aligned} \frac{\partial}{\partial t} F(x, t) = &\int_x^\infty du \int_0^u dy [F(y, t)\, F(u-y, t) - F(x, t)\, F(u-x, t)] \\ &- C_R F(x, t) \int_0^\infty F(y, t)\, dy - \hat{N}\hat{C}_R F(x, t) \end{aligned} \tag{9}$$

The removal collision frequencies C_R and $\hat{C}_R$ are positive constants.

2.2. General Solution of the VHP Equation with Removal

First, in order to determine the time-dependent particle number $N(t)$, we take the zeroth moment of Eq. (9) and obtain the differential equation

$$\frac{dN(t)}{dt} + C_R N(t)^2 + \hat{C}_R \hat{N} N(t) = 0 \tag{10}$$

Equation (10) is solved by the function

$$N(t) = \frac{N(0)\, \hat{N}\hat{C}_R}{[\hat{N}\hat{C}_R + N(0)\, C_R] \exp(\hat{N}\hat{C}_R t) - N(0)\, C_R} \tag{11}$$

For simplicity, we set $N(0) = 1$.

Next we calculate the first moment of Eq. (9), which results in the following equation for the total energy of the system:

$$\frac{dE(t)}{dt} + C_R E(t)\, N(t) + \hat{C}_R \hat{N} E(t) = 0 \tag{12}$$

The solution according to $E(0) = 1$ is given by

$$E(t) = \frac{\hat{N}\hat{C}_R}{(\hat{C}_R \hat{N} + C_R)\exp(\hat{N}\hat{C}_R t) - C_R} \tag{13}$$

Hence, we observe an identical time dependence for the total energy and the particle density, which may be interpreted as a constant temperature T during the time evolution of our particle system,

$$\frac{k}{m} T(t) = \frac{1}{N(t)} \int_0^\infty xF(x, t)\, dx = \text{const} \tag{14}$$

where k denotes the Boltzmann constant and m the particle mass.

Following Boffi and Spiga,[16] we look for solutions of Eq. (9) in the form $F(x, t) = N(t) \cdot V(x, t)$. Using this ansatz, we can transform Eq. (9) into

$$\begin{aligned} &\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{1}{N(t)} \frac{dN(t)}{dt} + C_R N(t)\, V(x, t) + \hat{C}_R \hat{N} V(x, t) \\ &\quad = N(t) \int_x^\infty du \int_0^u dy [V(y, t)\, V(u-y, t) - V(x, t)\, V(u-x, t)] \end{aligned} \tag{15}$$

Some manipulations on Eq. (11) lead to

$$\frac{1}{N(t)} \frac{dN(t)}{dt} = -C_R N(t) - \hat{C}_R \hat{N} \tag{16}$$

With this result in mind and performing the second integral term in Eq. (15), we obtain

$$\begin{aligned} &\frac{\partial V(x, t)}{\partial t} + E(t)\, V(x, t) + xN(t)\, V(x, t) \\ &\quad = N(t) \int_x^\infty du \int_0^u dy\, V(y, t)\, V(u-y, t) \end{aligned} \tag{17}$$

by taking into account Eqs. (11) and (13).

The introduction of a new independent time variable

$$\begin{aligned} \tau(t) &= \int_0^t dt'\, N(t') \\ &= \frac{1}{C_R} \ln \left\{ [\hat{C}_R \hat{N} + C_R - C_R \exp(-\hat{N}\hat{C}_R t)] \frac{1}{\hat{C}_R \hat{N}} \right\} \end{aligned} \tag{18}$$

reduces Eq. (17) to

$$\left(\frac{\partial}{\partial\tau}+x+1\right)V(x,\tau)=\int_x^\infty du\int_0^u dy\, V(u-y,\tau)\, V(y,\tau) \tag{19}$$

This integrodifferential equation for $V(x,\tau)$ is formally equivalent to the VHP-Boltzmann equation without removal terms (5). It can be solved for arbitrary initial distributions $F(x,0)=V(x,0)$ by applying the Laplace transformation

$$G(z,\tau)=\int_0^\infty dx\, e^{-xz}V(x,\tau)=L[V(x,\tau)] \tag{20}$$

as shown in ref. 8.

After some analysis, which is omitted here, one obtains the general solution of Eq. (19),

$$G(z,\tau)=\frac{\phi(z+\tau)+(z-1)\,e^{-\tau}}{(z+1)\,\phi(z+\tau)-e^{-\tau}} \tag{21}$$

The function $\phi(z)$ has to be determined from the Laplace transform of the initial distribution $V(x,0)=F(x,0)/N(0)$,

$$\phi(z)=\frac{G(z,0)+(z-1)}{G(z,0)(z+1)-1} \tag{22}$$

Finally, the time evolution of $V(x,\tau)$ is found via the inverse Laplace transformation

$$V(x,\tau)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, e^{xz}G(z,\tau) \tag{23}$$

where the path of integration should lie to the right of all points z where $G(z,\tau)$ is singular. Thus, for the distribution function we obtain $F(x,t)=N(t)\cdot V(x,\tau(t))$ under consideration of Eq. (18).

2.3. Asymptotic Properties of the Distribution Function

In the VHP model without removal interactions the Maxwell distribution e^{-x} is the equilibrium solution of the Boltzmann equation, a fact following from the validity of an H-theorem.

In our case, the time transformation (18) shows the asymptotic behavior

$$\lim_{t\to\infty}\tau(t)=\frac{1}{C_R}\ln\left[1+\frac{C_R}{\hat{N}\hat{C}_R}\right] \tag{24}$$

This means that the long-time characteristics of $V(x, \tau)$ are described by the distribution function

$$V(x, t=\infty)=V\left(x, \tau=\frac{1}{C_R}\ln\left[1+\frac{C_R}{\hat{N}\hat{C}_R}\right]\right) \tag{25}$$

depending on the removal collision frequencies C_R and $\hat{C}_R$ and the density of the host medium $\hat{N}$. As the independent time variable τ of $V(x, \tau)$ is finite, even in the case $t \to \infty$, $V(x, \tau)$ never approaches a Maxwellian. But, if we neglect removal events caused by the scattering of test particles with the host medium, the Maxwell distribution again describes the asymptotic behavior of $V(x, \tau)$, because

$$\lim_{\hat{N}\hat{C}_R \to 0} \lim_{t \to \infty} \tau(t) = \lim_{\hat{N}\hat{C}_R \to 0} \frac{1}{C_R}\ln\left[1+\frac{C_R}{\hat{C}_R\hat{N}}\right] \to \infty \tag{26}$$

We want to emphasize that in both cases the solution of Eq. (9) given by $F = V \cdot N$ obviously tends to zero for $t \to \infty$.

2.4. Strong Removal Interactions Between Test Particles and Field Particles

In the case of dominant removal interactions of the test particles with the host medium, i.e., $C_R \ll \hat{C}_R$ and $\hat{C}_R \gg 1$, we obtain instead of Eq. (9) the simple transport equation

$$\frac{\partial F(x, t)}{\partial t} = -\hat{C}_R \hat{N} F(x, t) \tag{27}$$

Its solution is given by

$$F(x, t) = F_0(x)\exp(-\hat{C}_R \hat{N} t) = F_0(x)\, N(t) \tag{28}$$

which means that the initial distribution decays proportional to $N(t)$.

The time-dependent particle number

$$N(t) = \exp(-\hat{C}_R \hat{N} t) \tag{29}$$

according to $N(0)=1$, has been calculated from the first moment of Eq. (27).

3. THE BOLTZMANN EQUATION IN THE VHP MODEL FOR A TAGGED PARTICLE

The nonlinearity on the right side of Eq. (4) may be suppressed if we set[17]

$$F(u-y, t)=\exp(-u+y), \qquad F(u-x, t)=\exp(-u+x) \tag{30}$$

Hence, we write the resulting linear VHP-Boltzmann equation in the following manner:

$$\left(\frac{\partial}{\partial t}+x+1\right) F_L(x, t)=\int_x^\infty du\, e^{-u} \int_0^u dy\, e^y F_L(y, t) \tag{31}$$

Equation (31) describes a particular particle in a bath of similar particles in thermal equilibrium.

Removal terms arise because of interactions between test particles and field particles of the same kind with the total density $\hat{N}_1$, and field particles of a host medium (total density $\hat{N}_2$)

$$\left(\frac{\partial F_L(x, t)}{\partial t}\right)_R=\hat{C}_{R1}\hat{N}_1 F_L(x, t)+\hat{C}_{R2}\hat{N}_2 F_L(x, t)=\hat{C}_R\hat{N}F_L(x, t) \tag{32}$$

Thus we obtain the linear VHP equation including removal terms in the form

$$\left(\frac{\partial}{\partial t}+x+1+\hat{C}_R\hat{N}\right) F_L(x, t)=\int_x^\infty du\, e^{-u} \int_0^u dy\, e^y F_L(y, t) \tag{33}$$

From Eq. (33), taking the zeroth and first moments, respectively, we can easily derive the expressions for the particle number and the total energy of the system

$$N(t)=E(t)=\exp(-\hat{C}_R\hat{N}t) \tag{34}$$

We set $F_L(x, t)=N(t)\cdot H_L(x, t)$ and Eq. (33) results in

$$\left(\frac{\partial}{\partial t}+x+1\right) H_L(x, t)=\int_x^\infty du\, e^{-u} \int_0^u dy\, e^y H_L(y, t) \tag{35}$$

It is remarkable to note that in this linear case the considered removal events do not affect the auxiliary distribution function $H_L(x, t)$.

Equation (35) is formally equal to Eq. (33) and can be solved in closed form by using the Laplace transformation[16]

$$G(z, t)=\frac{z(z+t+1)}{(z+1)(z+t)}e^{-t}\left(G_0(z+t)-\frac{1}{z+t+1}\right)+\frac{1}{z+1} \tag{36}$$

with $G_0(z)=L[F_L(x, 0)]$ and $G(z, t)=L[F_L(x, t)]$.

4. SOURCE TERMS IN THE VHP MODEL

Starting from Eq. (4), we add a general source term $Q(x, t) = S_0 \cdot S(x, t)$, $S_0 \geqslant 0$, and take the zeroth and first moments of the resulting transport equation. We find

$$\frac{dN(t)}{dt} = S_0 \int_0^\infty S(x, t)\, dx \tag{37}$$

and

$$\frac{dE(t)}{dt} = S_0 \int_0^\infty dx\, xS(x, t) \tag{38}$$

For physical reasons the right sides of Eqs. (37) and (38) should be finite. Hence we require

$$\int_0^\infty dx\, S(x, t) < \infty, \qquad \int_0^\infty dx\, xS(x, t) < \infty \tag{39}$$

The conditions (39) are fulfilled by the stationary δ-source $Q(x) = S_0 \delta(x - x_0)$, $x_0 > 0$, the Maxwell source $Q(x, t) = S_0 e^{-x}$, and a source term in the form

$$Q(x) = \left\{ \begin{array}{ll} S_0, & a \leqslant x \leqslant a + \varepsilon \\ 0, & \text{else} \end{array} \right\} \qquad a > 0, \quad \varepsilon > 0 \tag{40}$$

Including source as well as removal terms we obtain the following nonlinear VHP-Boltzmann equation:

$$\begin{aligned} &\frac{\partial}{\partial t} F(x, t) + C_R N(t)\, F(x, t) + \hat{C}_R \hat{N} F(x, t) \\ &\quad = \int_x^\infty du \int_0^u dy [F(y, t)\, F(u - y, t) \\ &\qquad - F(x, t)\, F(u - x, t)] + S_0 S(x) \end{aligned} \tag{41}$$

Taking the zeroth moment of Eq. (41) leads to the differential equation for the particle number $N(t)$,

$$\frac{dN(t)}{dt} + C_R N(t)^2 + \hat{C}_R \hat{N} N(t) = S_0 \tag{42}$$

Equation (42) is solved by the function

$$N(t)=\frac{1}{C_R}\left(\frac{\beta\left\{1+\dfrac{(C_R N_0+\frac{1}{2}\hat{C}_R\hat{N}+\beta)}{(C_R N_0+\frac{1}{2}\hat{C}_R\hat{N}-\beta)}\exp(2\beta t)\right\}}{\dfrac{(C_R N_0+\frac{1}{2}\hat{C}_R\hat{N}+\beta)}{(C_R N_0+\frac{1}{2}\hat{C}_R\hat{N}-\beta)}\exp(2\beta t)-1}-\frac{1}{2}\hat{C}_R\hat{N}\right) \tag{43}$$

with

$$\beta=[\tfrac{1}{4}(\hat{C}_R\hat{N})^2+S_0 C_R]^{1/2} \tag{44}$$

$N(t)$ shows the long-time behavior

$$\lim_{t\to\infty} N(t)=\frac{1}{C_R}\left[\beta-\frac{1}{2}\hat{C}_R\hat{N}\right]>0 \tag{45}$$

and, if no background removal effects occur,

$$\lim_{\hat{C}_R\to 0}\lim_{t\to\infty} N(t)=\left(\frac{S_0}{C_R}\right)^{1/2} \tag{46}$$

Equation (46) means that the equilibrium particle number depending only on the ratio between the source density S_0 and the collision frequency C_R is reached from above if the initial number of particles $N(0)$ is greater than $(S_0/C_R)^{1/2}$ and otherwise from below.

This result can be derived as well from Eq. (42) by neglecting the term $dN(t)/dt$ in the case of thermal equilibrium and setting $\hat{C}_R\hat{N}=0$. Thus, we obtain $C_R N(t)^2=S_0$ and therefore Eq. (46).

5. NUMERICAL RESULTS

5.1. Nonlinear VHP Model with Removal Interactions

We consider the initial distribution as a superposition of two Maxwellians,

$$F(x,0)=V(x,0)=\frac{v_1}{\tau_1}e^{-x/\tau_1}+\frac{v_2}{\tau_2}e^{-x/\tau_2} \tag{47}$$

For $F(x,0)$ to be positive we require $\tau_2<1$ and $\tau_1+\tau_2\geqslant 1$. The normalization of particle number and total energy of the system to unity implies

$$v_1=\frac{1-\tau_2}{\tau_1-\tau_2},\qquad v_2=\frac{\tau_1-1}{\tau_1-\tau_2} \tag{48}$$

Performing the Laplace transformation of the initial distribution (47),

$$G(z, 0) = \frac{\nu_1}{1 + \tau_1 z} + \frac{\nu_2}{1 + \tau_2 z} \tag{49}$$

and inserting this result into Eqs. (21) and (22), we obtain finally the Laplace-transformed distribution function for arbitrary times,

$$G(z, t) = \frac{N_1(\tau)}{1 + T_1(\tau)\, z} + \frac{N_2(\tau)}{1 + T_2(\tau)\, z} \tag{50}$$

Therefore the distribution function has the form

$$V(x, \tau) = \frac{N_1(\tau)}{T_1(\tau)}\, e^{-x/T_1(\tau)} + \frac{N_2(\tau)}{T_2(\tau)}\, e^{-x/T_2(\tau)} \tag{51}$$

As shown by Hendriks and Ernst,[8] the unknown functions $T_1(\tau)$, $T_2(\tau)$, $N_1(\tau)$, and $N_2(\tau)$ are solutions of the quadratic equation

$$\begin{aligned} &T^2(\tau_1 \tau_2(\tau - 1 + e^{-\tau}) + (\tau_1 + \tau_2)(1 - e^{-\tau}) + e^{-\tau}) \\ &\quad - T(\tau_1 \tau_2 \tau + \tau_1 + \tau_2) + \tau_1 \tau_2 = 0 \end{aligned} \tag{52}$$

and fulfill the relations

$$N_1(\tau) = \frac{1 - T_2(\tau)}{T_1(\tau) - T_2(\tau)}, \qquad N_2(\tau) = \frac{T_1(\tau) - 1}{T_1(\tau) - T_2(\tau)} \tag{53}$$

with respect to the time transformation $\tau = \tau(t)$ given in Eq. (18). For $\tau \to \infty$ it follows from Eqs. (52) and (53) that $T_1 \to 1$, $T_2 \to 0$ and $N_1 \to 1$,

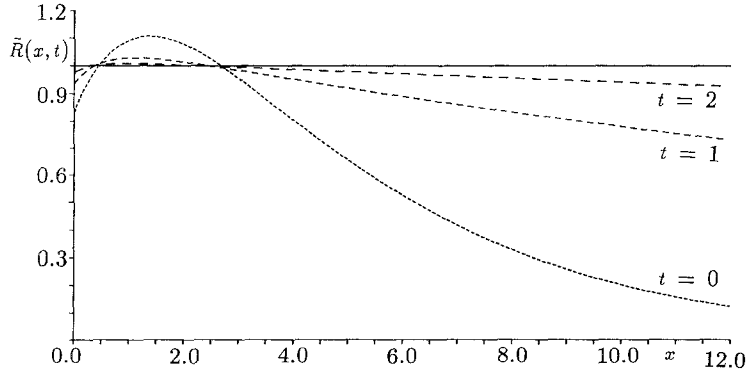

Fig. 1. Monotonic approach of $\tilde{R}(x, t) = V(x, t)/e^{-x}$ to equilibrium corresponding to initial condition (53) with $\tau_1 = 4/5$, $\tau_2 = 3/5$, and $C_R = \hat{C}_R = 0$.

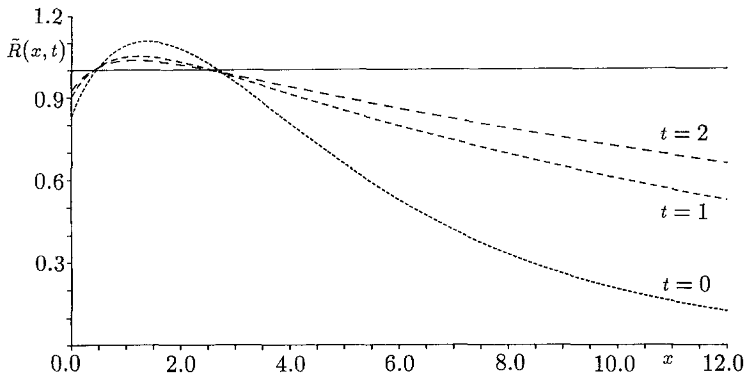

Fig. 2. Parameters as in Fig. 1, but $C_R = 2$.

$N_2 \to 0$. Inserting these long-time characteristics into Eq. (51), we get the Maxwell distribution.

Figures 1–3 show the convergence of the relative distribution $\tilde{R}(x, t) = V(x, t)/V(x, \infty)$ to the equilibrium state $\tilde{R}(x, \infty) = 1$.

Comparing Figs. 1 and 2, we observe a delayed relaxation in the case that removal events are taken into account (Fig. 2). The mathematical reason is an extension of the time base $\tau(t) < t$ according to Eq. (18) for $\hat{N}\hat{C}_R = 0$. Physically, we can argue that a slower relaxation of the distribution function to the equilibrium state is caused by the loss of particles.

In the case $\hat{C}_R = 2$ (Fig. 3) we obtain from Eqs. (51)–(53) and (25) an equilibrium distribution in the form

$$V(x, \infty) = 1.3288 e^{-1.1013x} - 0.4465 e^{-2.1617x} \tag{54}$$

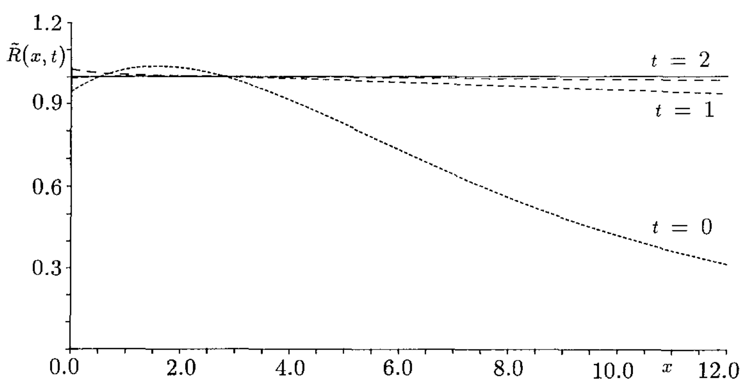

Fig. 3. Relative distribution function $\tilde{R}(x, t) = V(x, t)/V(x, \infty)$ for $\tau_1 = 4/5$, $\tau_2 = 3/5$, $C_R = \hat{C}_R = 2$, and $\hat{N} = 1$. Initial state as in Fig. 1.

which widely diverges from the Maxwell distribution e^{-x}. This affects a deviation of the relative initial distribution $\tilde{R}(x, 0)$ in comparison to Figs. 1 and 2. Contrary to the first impression, the relaxation only seems to be faster in Fig. 3, because the initial state is already closer to the equilibrium. But the relative equilibrium distribution $\tilde{R}(x, \infty)=1$ is reached approximately earlier in time.

5.2. VHP Model with Removal Terms for a Tagged Particle

The Laplace-transformed initial distribution (49) is inserted into Eq. (36). The inverse Laplace transformations that appear can be carried out in closed form.[18] We get for $t \neq 1$

$$\begin{aligned} F_L(x, t) &= e^{-x} + e^{-x}\frac{e^{-t}}{t-1} \\ &\quad + \frac{te^{-t}e^{-x}}{t-1}\left(\frac{-\nu_1/\tau_1}{t+1/\tau_1-1} - \frac{\nu_2/\tau_2}{t+1/\tau_2-1}\right) \\ &\quad + e^{-t}\left[\frac{\nu_1}{\tau_1}e^{-x(t+1/\tau_1)} - \frac{\nu_1 t}{t+1/\tau_1-1}e^{-x(t+1/\tau_1)}\right] \\ &\quad + e^{-t}\left[\frac{\nu_2}{\tau_2}e^{-x(t+1/\tau_2)} - \frac{\nu_2 t}{t+1/\tau_2-1}e^{-x(t+1/\tau_2)}\right] \end{aligned} \tag{55}$$

and for $t=1$

$$\begin{aligned} F_L(x, 1) &= e^{-x} + \frac{1}{e}\nu_1\tau_1\left(\frac{1}{\tau_1^2}-1\right)e^{-x(1+1/\tau_1)} \\ &\quad + \frac{1}{e}\nu_2\tau_2\left(\frac{1}{\tau_2^2}-1\right)e^{-x(1+1/\tau_2)} \end{aligned} \tag{56}$$

We obtain for arbitrary energy x

$$\lim_{t\to 1} F_L(x, t) = F_L(x, 1) \tag{57}$$

and

$$\lim_{t\to 1-1/\tau_1} F_L(x, t) = M < \infty, \qquad \tau_1 > 1 \tag{58}$$

which prove the continuity of the distribution function for critical values of the time variable.

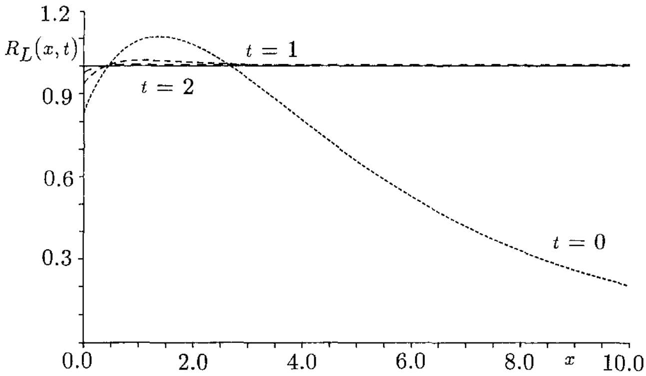

Fig. 4. Solutions of the linear Boltzmann equation. Initial state and parameters as in Fig. 3.

A comparison between Figs. 1 and 4 shows that this linear VHP model does not lead to a satisfactory description of the high-energy tail of the distribution function. This result makes clear the necessity of a nonlinear treatment of the VHP-Boltzmann equation in general.

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